University of Oregon Math 111 Notes

Center of Multicultural Academic Excellence

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Abstract

The Center of Multicultural Academic Excellence has created notes for Math 111 to support students of color to receive academic guidance and help in Math.

Contents

1 Introduction to Functions

Definition 1. A **function** is a relationship involving inputs and outputs. In order to be a function, every input must have exactly ONE output. f(input)= output

Definition 2. A **domain** is the set of inputs in the function (ie. all x's in the functions). The set of outputs is the **range** or **image** (ie. all y's in the functions).

Example: Let's say the grocery store is the function of buying items. If 10 people go to the grocery store and buy a certain amount of food, the domain is the amount of people going to the grocery store. The range or image is the amount of items each person bought.

Definition 3. **Function notation** is a formula written as Q = f(t) (ie. "Q is a function of t"). A **dependent variable** is a variable that will change based on what you input. Usually the output of a function. A **independent variable** does not change based on other factors. Usually an input of a function. Ex. Q changes when t is changed. Whereas, t is an independent variable because t doesn't change.

Example: Let's say $Q = f(t) = 2t + 12$ if t = 1 then $Q = f(1) = 2(1) + 12 = 14$. The independent variable is t=1, the dependent variable is Q=14.

Definition 4. A **mathematical domain** is used when there is not domain specified, it is the largest set of real numbers used as inputs in the functions.

Example: The domain of a function $f(x) = x$ is all real numbers (ie. $(-\infty, \infty)$; meaning any number you input for x is in the domain). However if the domain was specified, $x > -2$, then the domain would be $(-2, \infty)$.

Definition 5. A **piece-wise function** is a function that has different portions/ pieces in the domain.

Example: Given the function $f(x) = \{x^2$ if $x \geq 0$, x if $x < 0\}$.
If x < 0 then the function used is x. If $x \geq 0$ then the function used is x^2. You plug in values into a certain equation based on what x is.

Functions by Table:
The domain of a function in table format is usually the left side of the table. And the range is the right side.

Example: Given the table:

t	q=F(t)
-3	-1
-1	5
2	19
5	7
8	-1

The domain of q is [-3,8]. The range of the function q is [-1,19].

2 Linear Functions and Average Rate of Change

Definition 6. The **average rate of change** is the average change in a function. Given by the equation.

$$ARC = \frac{f(b)-f(a)}{b-a} \text{ or } \frac{\Delta Q}{\Delta t}$$

In words, ΔQ is the change in y value, and Δt is the change in x values (usually time).

Example: Q= f(t)= t^3. If given the interval [0,3], we can compute the average rate of change.

$$\frac{\Delta Q}{\Delta t} = \frac{f(3) - f(0)}{3 - 0} = \frac{3^3 - 0^3}{3 - 0} = \frac{27 - 0}{3 - 0} = \frac{27}{3} = 9$$

The average rate of change is 9.

Definition 7. A function is **strictly increasing** on an interval if f(b) > f(a). Meaning if y_2 is larger than y_1. On a graph, a function is increasing if it rises to the top right corner.
A function is **strictly decreasing** on an interval if f(b) < f(a). Meaning if y_2 is smaller than y_1. On a graph, a function is decreasing if it falls to the lower left corner.
A function is **constant** on an interval if f(b) = f(a). Meaning if y_2 is equal to y_1. On a graph, a function is constant if it is a horizontal line.

If the average rate of change is positive, then the function is increasing.
If the average rate of change is negative, then the function is decreasing.
If the average rate of change is 0, then the function is constant.

Definition 8. A **linear function** is a function f, that between 2 points on the graph of the function, is the average rate of change.

$$f(x) = mx + b$$

The variable m is the slope or average rate of change and the variable b is a constant.
Slope can be found by taking the average rate of change between two points.

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: If given two points on a graph, (0,2) and (2,8), we can find the slope (average rate of change), constant b, and the linear function.

$$slope = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 2}{2 - 0} = \frac{6}{2} = 3$$

The variable b represents a constant which can also be referred to as when x=0 or the y-intercept. In this case, x=0 when y=2. Therefore, our constant b=2.
Now that we know the slope and constant we can plug that into the linear function equation of $f(x) = mx + b$. Our equation would be $f(x) = 3x + 2$.
To double check out work, we can plug in the $x_1 and x_2$ values to make sure we get the correct points.

$$f(0) = 3(0) + 2 = 0 + 2 = 2$$

$$f(2) = 3(2) + 2 = 6 + 2 = 8$$

Since we get the points (0,2) and (2,8), our linear equation is correct.

Theorem: If a slope is positive then the function is strictly increasing along the domain.
If a slope is negative then the function is strictly decreasing along the domain.
If a slope is 0 then the function is constant along the domain.

Theorem: A linear graph will always be a line. This is because of the average rate of change (slope) is constant between each set of points. This is when there is x^1 or x. x^2 or x^3 are not linear functions, only x represents a linear function.

Definition 9. A set of n points is **collinear** if there is a single vertical line that goes through all n points.

3 Quadratic Functions and Higher-Degree Polynomials

Definition 10. A **quadratic function** is a function which can be written in the form

$$f(t) = at^2 + bt + c$$

a, b and c are constants.

Definition 11. If a<b<c in $f(t) = at^2 + bt + c$, then the **second difference** can be computed with the equation

$$[f(c) - f(b)] - [f(b) - f(a)]$$

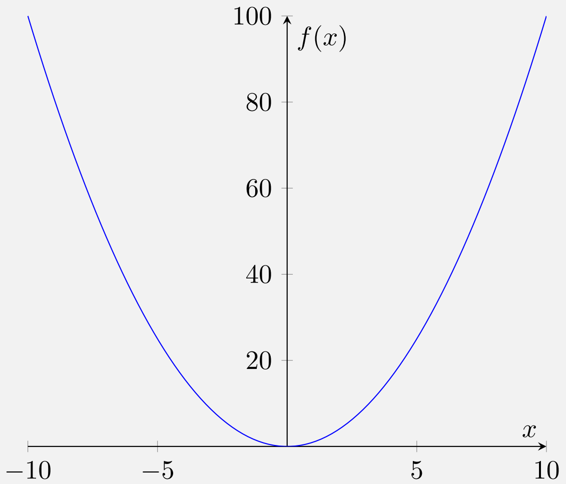
Theorem:
The maximum or minimum of a quadratic function can be determined using the extremum formula

$$t = \frac{-b}{2a}$$

. If a<0, then the function has a maximum. If a>0, then the function has a minimum.
Example:
Given the equation $f(t) = 4.9t^2 + 5.17t + 10, a = -4.9, b = 5.17, c = 10$.
We can plug this into our extremum formula.

$$t = \frac{-b}{2a} = \frac{-5.17}{2(-4.9)} = 0.528$$

Definition 12. A parabola is a common type of quadratic equation.



A **positive curvature** looks like the graph above, like U. The vertex will be the lowest point on the graph. A **negative curvature** looks like a hump. The vertex will be the highest point on the graph.

Theorem:
The Vertex Form of a Quadratic is written in the form

$$Q = f(t) = a(t - t_{vert})^2 + Q_{vert}.$$

The maximum or minimum points (the vertex) are the points (t_{vert}, Q_{vert}).

Example:
Say we were given the vertex as (1,3) and a = 2, then the Vertex Form of a Quadratic would be

$$Q = f(t) = 2(t - 1)^2 + 3$$

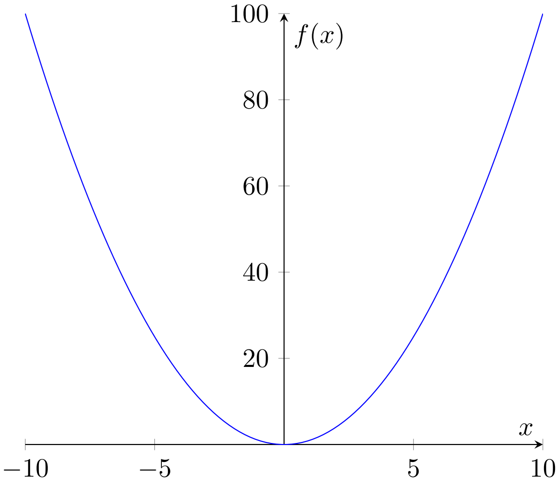
Definition 13. A **monomial** is an expression of the form at^b, where a is any constant (called its coefficient) and b is a non-negative whole number (called its degree).
A **polynomial** is a sum or difference of any number of monomials (including just one).
A **polynomial function** is a function whose formula can be written as a polynomial.
The **leading term** of a polynomial is the term containing the highest power on t. This highest power is the **degree** of the polynomial.
The **leading coefficient** of a polynomial is the coefficient of the leading term.

Example:
Given the function $f(t) = 3x - 1$
The degree = 1, since x can also be written as x^1.
The leading coefficient = 3, its the coefficient in front of the highest degree x variable.
The leading term = 3x, both the coefficient and x variable.

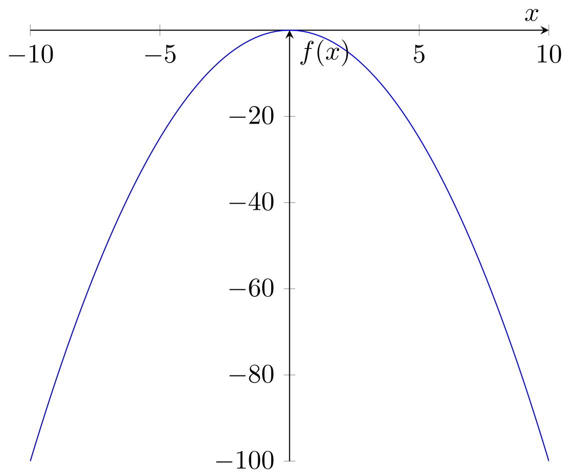
Example:
Given the function $f(t) = 3x^2 - x + 1$
The degree = 2, since x^2 is the highest value x.
The leading coefficient = 3, its the coefficient in front of the highest degree x variable.
The leading term = $3x^2$, both the coefficient and x variable.

Definition 14. (Long-Term Behavior of Polynomials)
If the leading coefficient is positive then the long term behavior goes towards ∞ as t increases.
If the leading coefficient is negative then the long term behavior goes towards $-\infty$ as t increases.

Example:



In the graph above of x^2, we can find the leading coefficient is $+1$. Therefore the graph goes toward ∞.



In the second graph above of $-x^2$, we can find the leading coefficient is -1. Therefore the graph goes toward $-\infty$.

4 Rational Functions

Definition 15. The **reciprocal function** of t in defined to be

$$Q = f(t) = \frac{1}{t}$$

The **reciprocal square function** of t in defined to be

$$Q = f(t) = \frac{1}{t^2}$$

Example:
Consider the function $f(x) = 2x - 1$. The reciprocal function of f would be as follows:

$$\frac{1}{f(x)} = \frac{1}{(2x-1)}$$

Definition 16. A **rational function** is a function which can be written in the form

$$f(t) = \frac{p(t)}{q(t)}$$

where p(t) and q(t) are each polynomial functions (and q(t) $\neq$ 0).

Example:
Given $f(t) = \frac{3+x}{x^2-4x+7}$ is a rational function.
We can separate this into two polynomials, $p(x) = 3+x$ with leading term x; and $q(x) = x^2-4x+7$, with a leading term x^2.

Theorem: Long-Term Behavior of Basic Rational Functions

- For any constant k and positive whole number p, we write,

  $$\text{As t} \to \pm\infty\text{, then } \frac{k}{t^p} \to \ 0$$

  In words, if the bottom of the fraction is larger than the top of the fraction our answer will get closer and closer to zero.

  For example:
  If $f(t) = \frac{1}{t^2}$, then as t $\to \infty$ the denominator will become a really big number, and 1/(really big number) =0.

  *Side note: If you don't understand how a function can approach infinity, think of infinity as a really big number like 100,000. For the example above, if we had 1/100,000 = 0.00001. Which means your answer is super small and basically as we approach infinity (keep dividing by a bigger and bigger number) our answer will get smaller and smaller and closer to 0. So our answer is 0.

- For any positive real number k and integer p, we write

  $$\begin{gathered}\text{As t} \to \ 0\text{, with t} > 0\text{, then } \frac{k}{t^p} \to \infty \\ \text{As t} \to \ 0\text{, with t} < 0\text{, then } \frac{k}{t^p} \to -\infty \\ ,\end{gathered}$$

  In words, if you make the bottom of a fraction a tiny number, the whole thing gets larger and larger (either in the positive or negative direction).

  For example:
  If $f(t) = \frac{1}{t^2}$, then as t $\to$ 0 the denominator will become a really small number, and 1/(really small number) = $\pm\infty$.

Theorem: Long-Term Behavior of a General Rational Function

Let $f(t) = \frac{p(t)}{q(t)}$ be a rational function, where p and q are polynomial functions with leading terms P(t) and Q(t), respectively.
Then the long-term behavior of f(t) is the long-term behavior of the simplified function $\frac{P(t)}{Q(t)}$.
Example:
If given the equation $f(t) = \frac{3-t^2}{2t^2+1}$. Our leading terms are $P(t) = -t^2$ and $Q(t) = 2t^2$. Putting this in the long-term behavior of the simplified function above

$$\frac{P(t)}{Q(t)} = \frac{-t^2}{2t^2} = \frac{-1}{2}$$

So the long term behavior is $\frac{-1}{2}$. This only works if both leading terms are the same degree.

5 Exponential Functions

Definition 17. The **percent change** in a function f is on the interval [c,d] is

$$PC_{[c,d]} = \frac{f(d) - f(c)}{f(c)} \times 100\%$$

Example:
Find the percent change of the function $f(t) = 6 - 4t$ on the interval [1,3].
1. First we mist find out f(1) and f(3).
$f(1) = 6 - 4(1) = 6 - 4 = 2$
$f(3) = 6 - 4(3) = 6 - 12 = -6$
2. Now we plug into our equation.

$$PC_{[1,3]} = \frac{f(3) - f(1)}{f(3)} \times 100\% = \frac{-6 - (2)}{2} \times 100\% = \frac{-8}{2} \times 100\% = -4 \times 100\% = -400\%$$

Definition 18. An **exponential function** is a function which can be written in the form

$$f(t) = a \times (1 + r)^t$$

for a constant r called the function's **relative growth rate**. Another version of this formula is

$$f(t) = a \times b^t$$

with b = 1 + r. We need a to be positive and r > −1 (so then b must be positive).
The variable b can also be referred to as the constant **growth rate**.
An exponential function f (t) has the property that for every one-unit increase in t, f (t) is multiplied by a factor

Example:
Given the exponential function $f(t) = 3 \times (1.2)^t$ has a growth factor or b = 1.2. Since b = 1+r then 1.2 = 1+r and r = 0.2 or 20%. Therefore, f(t) has a growth rate of 20%.

Theorem: Exponential Function as Proportional Change

If Q is changing at a rate proportional to itself, so that R(t) = k · Q, where R is the rate of growth in Q and k is the **continuous growth rate**, then

$$Q = f(t) = ae^{kt}$$

where a is a positive constant (which is also the value of Q at t = 0).

Definition 19.

1. An exponential function written $f(t) = ab^t$ with b > 1, or also written as, $f(t) = ae^{kt}$ with k > 0, is a strictly increasing function and is said to exhibit **exponential growth**.

2. An exponential function written $f(t) = ab^t$ with 0 < b < 1, or also written as, $f(t) = ae^{kt}$ with k < 0, is a strictly decreasing function and is said to exhibit **exponential decay**.

Example:
If given the function $f(t) = 3 \times (1.2)^t$. This is an increasing function since b=1.2 which is greater than 1. The function is exponentially growing.
If given the function $f(t) = 3 \times e^{-2t}$. This is an decreasing function since k=-2 which is less than 0. The function is exponentially decaying.

You can always double check by graphing the function.

Theorem: Long-Term Behavior of Exponential Functions

The long-term behavior of an exponential function $f(t) = ab^t or f(t) = ae^{kt} is \infty$ if k>0 or b>1.
The long-term behavior of an exponential function $f(t) = ab^t or f(t) = ae^{kt} is 0$ if k < 0 or 0 < b < 1

Example:
If given the function $f(t) = 3 \times (1.2)^t$. The long term behavior is ∞.
If given the function $f(t) = 3 \times e^{-2t}$. The long term behavior is 0.

Theorem: Mathematical Domain and Image of an Exponential Function

For a function f (t) = abt or f (t) = aek t, with a > 0, we have that

Domain: $(-\infty, \infty)$ Image: $(0, \infty$

Example:
If given the function $f(t) = 3 \times e^{-2t}$. The domain of the function would be $(-\infty, \infty)$, since a=3 >0.

6 Power and Logarithmic Functions

Definition 20. The **logarithm with base** b is the value L so that $b^L = Q$. The logarithmic equivalent to this equation is written $L = log_b(Q)$ and read "log, base b, of Q".
The logarithm with base e, namely $log_e(Q)$, is commonly abbreviated $ln(Q)$. This is typically called the **natural logarithm**.

Example:
If given the equation $3^x = 8$, we can rewrite it as $x = log_3(8)$
Similarly, ln(v) = 5 can be rewritten v = e^5

Definition 21. A logarithmic function is a function which can be written in the form

$$f(t) = a + log_b(t)$$

,with positive $b \neq 1$. It is defined only for $t > 0$.

Definition 22. A logarithmic function $f(t) = a + log_b(t)$ with b > 1 is a strictly increasing function and is said to exhibit logarithmic growth.
A logarithmic function $f(t) = a + log_b(t)$ with 0 < b < 1 is a strictly decreasing function and is said to exhibit logarithmic decay.

Theorem: Properties of Logarithms

For positive real numbers t, u, and b 6= 1, and any real number n, we have

The Sum Property: $log_b(t) + log_b(u) = log_b(t \times u)$
The Difference Property: $log_b(t) - log_b(u) = log_b(\frac{t}{u})$
The Constant Multiple Property: $n \times log_b(t) = log_b(t^n)$

Theorem: Change of Base Formula

As long as $b \neq 1$ and Q are positive, the expression $log_b(Q)$ can be rewritten as a ratio of two other logarithms:

$$log_b(Q) = \frac{ln(Q)}{ln(b)}$$

Example:
Given the expression $log_1.04(2)$ can be rewritten as

$$log_{1.04}(2) = \frac{ln(2)}{ln(1.04)} \approx 17.6830$$

Definition 23. A **power function** is a function of the form

$$Q = f(t) = at^b$$

for constants a (positive) and b.

- A power function is also polynomial if b is a non-negative whole number.
- A power function is rational if b is a negative integer.
- The domain of a non-polynomial power function includes $[0, \infty]$ for [b>0]. For some non-integer values of b, the domain also includes the interval $(-\infty, 0)$.
- The domain of a non-polynomial power function includes $(0, \infty)$ for [b<0]. For some non-integer values of b, the domain also includes the interval $(-\infty, 0)$.

7 Composition and Arithmetic of Functions

Definition 24. For two functions, f(t) and g(t), for any t in the domain of both f and g, we can perform the following **arithmetic operations on functions**:

- $$(f + g)(t) = f(t) + g(t)$$
- $$(f - g)(t) = f(t) - g(t)$$
- $$(f \times g)(t) = f(t) \times g(t)$$
- $$(\frac{f}{g})(t) = \frac{f(t)}{g(t)}$$

The domains of $f + g, fg$, and $f \times g$ are simply the intersection of the domain of f and the domain of g. The domain of $\frac{f}{g}$ is the intersection of the domains of f and g, excluding t such that g(t) = 0.

Example:
If f(2) = 4 and g(2) = 7, then (f + g)(2) = f(2) + g(2) = 4 + 7 = 11.
If f(2) = 4 and g(2) = 7, then (f - g)(2) = f(2) - g(2) = 4 - 7 = -3.
If f(2) = 4 and g(2) = 7, then $(f \times g)(2) = f(2) \times g(2) = 4 \times 7 = 28$.
If f(2) = 4 and g(2) = 7, then $\frac{f}{g}(2) = \frac{f(2)}{g(2)} = \frac{4}{7}$

Definition 25. The **composition of f with g** is defined to be f[g(t)], sometimes written (f ∘ g)(t), and read "f of g of t" or "f composed with g of t". You might also read this as "f after g", to remind you of the order in which the input is placed.

$$(f \circ g)(t) = f[g(t)]$$
$$(g \circ f)(t) = g[f(t)]$$
$$(f \circ f)(t) = f[f(t)]$$
$$(g \circ g)(t) = g[g(t)]$$

Example:
Let $f(x) = 3t + 1$ and $g(t) = t^2 1$.
$(f \circ g)(t) = f[g(t)] = f(t^2 - 1) = 3(t^2 - 1) + 1 = 3t^2 - 3 + 1 = \underline{3t^2 - 2}$

$(g \circ f)(t) = g[f(t)] = g(3t + 1) = (3t + 1)^2 - 1 = 9t^2 + 6t + 1 - 1 = \underline{9t^2 + 6t}$

$(f \circ f)(t) = f[f(t)] = f(3t + 1) = 3(3t + 1) + 1 = 9t + 3 + 1 = \underline{9t + 4}$

$(g \circ g)(t) = g[g(t)] = g(t^2 - 1) = (t^2 - 1)^2 - 1 = t^4 - 2t^2 + 1 - 1 = \underline{t^4 - 2t^2}$

Definition 26. The **domain of the composite function** f ∘ g is the set of all elements in the domain of g such that the image of each element is also in the domain of f. In other words, we would check each number, a, in the domain of g to see if g(a) is in the domain of f . If it is, then a is part of the domain of f ∘ g.

8 Function Inverses

Definition 27. The **inverse of a function** f , if it exists, is the function we'll call $f^[1]$ whose rule is that for any t in the domain of f and Q in the image of f , whenever Q = f(t), then

$$f^{[-1]}(Q) = t$$

This definition has the effect of implying that

$$(f^{[-1]} \circ f)(t) = t$$

and

$$(f \circ f^{[-1]})(Q) = Q$$

Theorem: Exponential and Logarithmic Functions are Inverses

For $f(t) = b^t$ we have

$$f^{[-1]}(Q) = log_b(Q).$$

Relatedly, for $g(t) = log_b(t)$, we have

$$g^{[-1]}(Q) = b^Q$$

Example:
Let $f(t) = 2^t$. Then

$$f^{[-1]}(Q) = log_2(Q).$$

We can see this work in example, at least, by seeing that $f(0) = 2^0 = 1$, and then taking that output

$$f^{[-1]}(1) = log_2(1) = 0.$$

Definition 28. A function f(t) is **one-to-one** if, for each value Q in the image of f , there is exactly one t in the domain of f so that Q = f(t).
Another way to say that f is one-to-one is to say that it has an inverse (or that it is **invertible**).

Theorem: Finding an Inverse

The general process for finding the inverse of a function Q = f(t) is to exchange the roles of t and Q: that is, make t the "output" and Q the "input".

Example:
Let $g(t) = 3 - t^3$ and $Q = 3 - t^3$. To find the inverse solve for t.

$$Q = g(t)$$
$$Q = 3 - t^3$$
$$t^3 = 3 - Q$$
$$\underline{t = \sqrt[3]{3 - Q}}$$

Theorem: Domain and Image of an Inverse Function

For a one-to-one function f with domain D and image C, the inverse function $f^{[1]}$ has domain C and image D.

Theorem: Graph of an Inverse Function

Given a one-to-one function Q = f(t), the graph of $Q = f^{[1]}(t)$ is the graph of Q = f(t) reflected about the line Q = t.